Algebra Instruction and Intervention

PaTTAN AYP Conference
July 13, 2011

Brad Witzel, PhD
Winthrop University
The 2005 & 2007 National Assessment of Educational Progress (NAEP) reported:

– 15% of Grade 4 students scored below the basic level
– 25% of Grade 8 students scored below the basic level
– 36% of Grade 12 students scored below the basic level
The 2005 & 2007 National Assessment of Educational Progress (NAEP) reported:

- 40% of Grade 4 students with disabilities scored below the basic level
- 66% of Grade 8 students with disabilities scored below the basic level
- 83% of Grade 12 students with disabilities scored below the basic level
Mathematics Performance (from Riccomini)

Translated to Real World Performance

• 78% of adults cannot explain how to compute interest paid on a loan
• 71% cannot calculate miles per gallon
• 58% cannot calculate a 10% tip
• 27% of 8th graders could not correctly shade 1/3 of a rectangle
• 45% could not solve a word problem that required dividing fractions

Teaching beyond grade and course

• Your responsibility for what students learn in your course implies that you are responsible for what they learned before your course.

• Students have been introduced and presented a lot of math information. However, every year teachers blame teachers from the year before for not preparing their students.

• Not only must you know your own course content, you must be aware of previous grade level content and what is expected in the following year.
What we are doing today:

• Learning the necessary foundational skills for algebra success
• Discussing research-supported methods for teaching foundational skills
• Replicating research-based and research-supported instruction in algebra
I hope that you leave affected.
NMP quotes

• “Few curricula in the United States provide sufficient practice to ensure fast and efficient solving of basic fact combinations and execution of the standard algorithms” (p. 26).
• “…students should be able to proceed successfully at least through the content of Algebra II…” (p. 15)
• “Teachers should recognize that from early childhood through elementary school years, the spatial visualization skills needed for learning geometry have already begun to develop. In contrast to the claims of Piagetian theory, young children appear to possess at least an implicit understanding of basic facts of Euclidean concepts. However formal instruction is necessary to ensure that children build upon this knowledge to learn geometry” (p. 29)
More NMP quotes

• “Differences in teachers account for 12% to 14% of total variability in students’ mathematics achievement gains” (p. 35).
• “Calculators should not be used on test items designed to assess computational facility” (p. 61).
• “Publishers should make every effort to produce much shorter and more focused textbooks” (p. xxiv).
• Paraphrase - Students need clear models with think alouds, many examples and opportunities for practice, and frequent feedback. (p. 48)
• More rigorous research for this group of students is needed (p. 49).
Algebra and CCSS – just a reminder

8th Grade

The Number System
• Know that there are numbers that are not rational, and approximate them by rational numbers.

Expressions and Equations
• Work with radicals and integer exponents.
• Understand the connections between proportional relationships, lines, and linear equations.
• Analyze and solve linear equations and pairs of simultaneous linear equations.

Functions
• Define, evaluate, and compare functions.
• Use functions to model relationships between quantities.

Geometry
• Understand congruence and similarity using physical models, transparencies, or geometry software.
• Understand and apply the Pythagorean Theorem.
• Solve real-world and mathematical problems involving volume of cylinders, cones and spheres.

Statistics and Probability
• Investigate patterns of association in bivariate data.

Algebra

Seeing Structure in Expressions
• Interpret the structure of expressions
• Write expressions in equivalent forms to solve problems

Arithmetic with Polynomials and Rational Expressions
• Perform arithmetic operations on polynomials
• Understand the relationship between zeros and factors of polynomials
• Use polynomial identities to solve problems
• Rewrite rational expressions

Creating Equations
• Create equations that describe numbers or relationships

Reasoning with Equations and Inequalities
• Understand solving equations as a process of reasoning and explain the reasoning
• Solve equations and inequalities in one variable
• Solve systems of equations
• Represent and solve equations and inequalities graphically

© Witzel, 2011
What troubles have you noticed?

• Curricular?

• Instructional?

• Familial or Community?
Nationally, what do algebra teachers say? (NMP, 2008)

743 algebra teachers in 310 schools nationally responded to a survey on algebra instruction and student learning in 2007.

Findings:

• The teachers generally rated their students’ background preparation for Algebra I as weak. The three skill areas in which teachers reported their students have the poorest preparation are rational numbers, word problems, and study habits.

• Regarding the best means of preparing students, 578 suggested a greater focus on mastery of elementary mathematical concepts and skills.

• Teachers were less excited about how current textbook approaches meet the needs of diverse student populations.
More findings from the NSAT

• Use of calculators was quite mixed with 33% saying they never use them and 31% use them frequently (more than once a week)
• 60% use physical tools less than once a week and only 9% use them frequently
• 51% consider “mixed-ability” grouping to be a moderate or serious problem with instruction
• The greatest challenge to teachers was #1 – “working with unmotivated students.” This was chosen by 58% of the middle school teachers and 65% of the high school teachers. The next most frequent response was “making mathematics accessible and comprehensible to all my students,” selected by 14% of the middle school teachers and 9% of the high school teachers.
Do middle school math courses add up to algebra preparation?

Sanders, Riccomini, & Witzel, 2005

<table>
<thead>
<tr>
<th>Code</th>
<th>Category</th>
<th>Entering Math Tech 1</th>
<th>Entering Algebra 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DAPR</td>
<td>Data Analysis, Probability &amp; Statistics</td>
<td>39</td>
<td>85</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(46.4%)</td>
<td>(88.5%)</td>
</tr>
<tr>
<td>DECM</td>
<td>Decimals, their Operations and Applications: Percent</td>
<td>11</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(13.1%)</td>
<td>(66.7%)</td>
</tr>
<tr>
<td>EQTN</td>
<td>Simple Equations and Operations with Literal Symbols</td>
<td>35</td>
<td>80</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(41.7%)</td>
<td>(83.3%)</td>
</tr>
<tr>
<td>EXPS</td>
<td>Exponents and Square Roots; Scientific Notation</td>
<td>27</td>
<td>62</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(21.1%)</td>
<td>(64.6%)</td>
</tr>
<tr>
<td>FRAC</td>
<td>Fractions and their Applications</td>
<td>3</td>
<td>43</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3.6%)</td>
<td>(44.8%)</td>
</tr>
<tr>
<td>GMMS</td>
<td>Measurement of Geometrical Objects</td>
<td>20</td>
<td>56</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(23.8%)</td>
<td>(58.3%)</td>
</tr>
<tr>
<td>GRPH</td>
<td>Graphical Representation</td>
<td>13</td>
<td>59</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(15.5%)</td>
<td>(61.5%)</td>
</tr>
<tr>
<td>INTG</td>
<td>Integers, their Operations &amp; Applications</td>
<td>27</td>
<td>83</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(32.1%)</td>
<td>(86.5%)</td>
</tr>
<tr>
<td></td>
<td>Total Number of Students per course</td>
<td>84</td>
<td>96</td>
</tr>
</tbody>
</table>

© Witzel, 2011
So, Where are we exactly?
What constitutes good instruction for struggling students (Gersten, Chard, & Witzel, 2008)

- Model approaches to solving problems many times – both easy and difficult
- “Think aloud” and teach students to do the same
- Frequent practice
- Independent practice should include discriminatory problems
- Word problems and computation should be integrated with real and imagined problems
- Use visuals, such as CRA, to represent problems
# What have you implemented so far?


<table>
<thead>
<tr>
<th>Category</th>
<th>Recommendation</th>
<th>Yes</th>
<th>No</th>
<th>How</th>
</tr>
</thead>
<tbody>
<tr>
<td>Overall plan Assessment</td>
<td>Screening all students to identify those who need interventions</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Intervention content</td>
<td>Interventions that focus on whole numbers (K-5) and rational numbers (4-8)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Intervention instruction</td>
<td>Interventions are taught explicitly</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Intervention content</td>
<td>Structural word problem instruction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Intervention instruction</strong></td>
<td><strong>Interventions include visual representations</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Intervention content</td>
<td>Interventions include at least 10 minutes on fluent fact retrieval</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Assessment</td>
<td>Progress monitoring for those receiving interventions as well as those at-risk</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Intervention instruction</td>
<td>Motivational strategies for those in interventions</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

© Witzel, 2011
Using Visual Representations

- The National Math Panel (2008) concluded that use of visual representations, such as the CRA sequence of instruction is a powerful instructional tool.

- Highest effect sizes with secondary students were from CRA instruction (Gersten et al., 2009; Witzel, Mercer, & Miller, 2003; Witzel, 2005)

- The IES Math RtI Panel concurred that representations and levels of representations must be reviewed in general education instruction as well as interventions.
"It may be wrong, but it's how I feel."
How can the use of arrays progress across grade level standards from multiplication of single digits to multiple digits to decimals to fractions to polynomials?
Findings: Visuals and Graphic Depictions of Problems

• When teachers used graphic representations to demonstrate problems only, results were much less consistent.

• Visuals were not particularly useful unless students were provided opportunities to practice using them.

• Highest effect sizes were for CRA with clear and explicit stepwise consistency (Gersten et al., 2009; Witzel, Mercer, & Miller, 2003; Witzel, 2005)
## CRA (Gersten et al, p. 35)

### 3 + X = 7

<table>
<thead>
<tr>
<th>Solving the Equation with Concrete Manipulatives (Cups and Sticks)</th>
<th>Solving the Equation with Visual Representations of Cups and Sticks</th>
<th>Solving the Equation with Abstract Symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>A <img src="image1.png" alt="Image" /> + X = <img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /> + X = <img src="image4.png" alt="Image" /></td>
<td>3 + 1X = 7</td>
</tr>
<tr>
<td>B <img src="image5.png" alt="Image" /> - <img src="image6.png" alt="Image" /> = <img src="image7.png" alt="Image" /></td>
<td><img src="image8.png" alt="Image" /> - <img src="image9.png" alt="Image" /> = <img src="image10.png" alt="Image" /></td>
<td>-3 -3</td>
</tr>
<tr>
<td>C <img src="image11.png" alt="Image" /> = <img src="image12.png" alt="Image" /></td>
<td><img src="image13.png" alt="Image" /> = <img src="image14.png" alt="Image" /></td>
<td>1X = 4</td>
</tr>
<tr>
<td>D <img src="image15.png" alt="Image" /> = <img src="image16.png" alt="Image" /></td>
<td><img src="image17.png" alt="Image" /> = <img src="image18.png" alt="Image" /></td>
<td><img src="image19.png" alt="Image" /> = 4</td>
</tr>
<tr>
<td>E <img src="image20.png" alt="Image" /> = <img src="image21.png" alt="Image" /></td>
<td><img src="image22.png" alt="Image" /> = <img src="image23.png" alt="Image" /></td>
<td>X = 4</td>
</tr>
</tbody>
</table>

### Concrete Steps
A. 3 sticks plus one group of X equals 7 sticks
B. Subtract 3 sticks from each side of the equation
C. The equation now reads as one group of X equals 4 sticks
D. Divide each side of the equation by one group
E. One group of X is equal to four sticks (i.e., 1X/group = 4 sticks/group; 1X = 4 sticks)
CRA approach

- CRA is the Concrete to Representational to Abstract sequence of instruction.
- Three stages of learning
  - C = Learning through concrete hands-on manipulative objects
  - R = Learning through pictorial forms of the math skill
  - A = Learning through work with abstract (Arabic) notation
- www.rtitlc.org

© Witzel, 2011
Examples

• We must start with some of the basics. After all, Algebra is often viewed as glorified arithmetic.

• Let’s play!
Multisensory Algebra

• Computational Facts and Fact Families
• Rational Numbers
  – Negative Number calculations
  – Fractions
• Algebraic
  – Reducing Expressions
  – Inverse Operations
  – Transformational Equations

© Witzel, 2011
Progressions

Mixed to improper and back

• $\frac{8}{2}$

• $\frac{8}{3}$

• $\frac{13}{4}$
This mixed fraction as an improper fraction is $\frac{22}{5}$.

Why?

Tricks are not helpful to students with memory problems, think about the concept and purpose to the calculations

Say, “Four and two – fifths” “And says to add”

$\frac{4}{1} + \frac{2}{5} \text{ or } \frac{20}{5} + \frac{2}{5} = \frac{22}{5}$.

This is why you multiply the fraction’s denominator and then add the numerator.
Progressions
Division of Fractions

• Why is it that when you divide fractions, the answer is larger? Also, why do you invert and multiply?

• \( \frac{2}{3} \) divided by \( \frac{1}{4} \) = \( \frac{2}{3} \times \frac{4}{1} \) = \( \frac{8}{3} \)

\[
\frac{2/3}{1/4} = \frac{8/3}{4/4} = \frac{8/3}{1/1} = \frac{8}{3}
\]
Progressions

Why teach the basics correctly

Adding with unlike denominators

Division of fractions

\[
\frac{5}{3} + \frac{\frac{1}{y}}{y^2} = \frac{\left(\frac{5y}{y} + \frac{1}{y}\right)}{\left(\frac{3y^2}{y^2} + \frac{2}{y^2}\right)} = \frac{\frac{5y+1}{y}}{\frac{3y^2+2}{y^2}} = \frac{\frac{y(5y+1)}{3y^2+2}}{1} = \frac{5y^2+y}{3y^2+2}
\]
Observations of CRA

• What visual representations have you used like this?

• What is the strength of this approach?

• What should come after this approach? What is lacking?
Why would CRA be effective?

(Witzel, Riccomini, & Schneider, 2008)

• Multimodal forms of math acquisition to aid memory and retrieval
• Multiple learning styles are being met to aid relevance and motivation
• Meaningful manipulations of materials allow students to rationalize abstract mathematics
• Procedural accuracy; provides an alternative to algorithm memorization of math rules
• Transportable without concrete materials
Teach each CRA lesson to mastery

- Model and guide students in their use of manipulative objects and pictorial representations.
- Teach students step by step gradually introducing mathematical vocabulary. Allow students to name or invent their stepwise procedures within instruction.
- Move from concrete to representational to abstract learning levels only after students show accuracy without hesitations in manipulations or drawings.
- Assess each level of learning according to stepwise procedures. Take account of students who created different procedures.
Multisensory Algebra instruction

- CRA sequence allows hands-on and pictorial exploration of content
- Reinforces arithmetic while covering algebra
- Enforces the concepts within algebra while making the solution appear more available
- Instruction includes researched pedagogical steps as well as an advanced math model

© Witzel, 2011
The progression of fractions

- After teaching her students subtraction and negative integers through the fraction computation using sticks, Mrs. Straube was able to transition into more abstract terms.

1. 

\[
\begin{array}{c}
\frac{3}{8} - \frac{7}{8} = \frac{6}{8}\\
\frac{4}{8} - \frac{4}{8} = \frac{3}{8}
\end{array}
\]

2. 

\[
\begin{array}{c}
6 \frac{3}{8} - 2 \frac{7}{8} = 2 \frac{3}{8}\\
4 \frac{4}{8} - 3 \frac{8}{8} = 3 \frac{4}{8}
\end{array}
\]

This approach works for most computational practices based on its roots in place value and expanded notation.
Expansions on fractions using identity (Zeichner, 2006)

$$\frac{3 (5 + z)}{3 (x - 7)} = \frac{1}{3} \frac{(5 + z)}{(x - 7)}$$

$$\frac{7x}{12} \cdot \frac{12}{5} = \frac{7x}{5} \cdot \frac{12}{12} = \frac{7x \cdot 12}{5 \cdot 12}$$
Trigonometric ratios

(Willie Ware and Dee Miller)
How else can CRA and other visual representations be implemented in Tier 1 or intervention support?

- Fractions
- Decimals
- Graphing
- Equations
- Vocabulary
Research Support

- From research studies
- To statewide initiatives
- To individual classrooms

© Witzel, 2011
Multisensory Algebra success

The graph shows the pretest and posttest results for Multisensory Algebra (Multi Alg) and traditional methods. The graph indicates a significant improvement in scores for the Multisensory Algebra group compared to the traditional group.

The bar chart compares the performance in Stanines 1-3, 4-6, and 7-9. The Multisensory Algebra group shows a higher number of students in the higher Stanines (7-9), indicating better overall performance.

© Witzel, 2011
Research Support

• Statistic
  • Students with learning difficulties using this model outperformed peers on posttest and follow-up measures (F=13.89, p<0.000) (Witzel, Mercer, & Miller, 2003)
  • Students with a history of high math achievement scores also show benefit on the posttest (F=10.37, p<0.01) and the follow-up (F=6.97, p<0.01) despite pretest favoring of traditional (F=12.18, p<0.001) (Witzel, 2005).

• Testimonial
  • Teachers wanted to stop using their current instructional series and textbooks
  • One teacher claimed that he would never teach algebra using any other method than through this model

More detailed statistical description available
FL scores from the Algebra Success Keys Project

(headed by Dr. Mary Little, PhD, UCF)

• In a FL DOE 2006 call of the critical need to improve student rates of learning where only 4 of 67 districts met AYP in math for students with disabilities, CRA gained much attention for it’s potential.
• 5 districts created assessments based on their states benchmarks that mirrored their statewide exams in order to test this model with their teachers.
• The results...
He stated that FCAT scores increased and number sense jumped levels to proficient after one year. This was new for this population.
Another math director

This director taught a first year teacher who said she was struggling reaching her children how to implement CRA in her Geometry classes.
A third director’s pilot looks at FCAT results

Number of Students out of 19

- lower
- higher
- same

© Witzel, 2011
Next Steps with Visual Representations

• Working with manipulatives
  Manipulative objects do not teach children, teachers do. The manipulatives are mere tools to reach an outcome, usually an abstract one.

• Organize the sequence of the instructional steps

• Practice math dialogue to match instructional procedures.
Choosing Manipulatives and Preparing for Instruction

• Review the abstract problem solving steps and processes before choosing manipulative objects

• Ask yourself:
  1. Are these manipulatives easy to use?
  2. How can these manipulatives be used for concept and process?
  3. Can I follow the same abstract steps using these manipulatives?
  4. How will the pictorial representation stage appear with these manipulatives?
  5. How many similar math skills can be taught using these manipulatives?
Some Tips for Concrete Instruction

• Concrete objects must be demonstrated
• Students should be allowed to explore math principles after knowing how to use them
• Concrete does not replace teaching, it requires more preparation
• Use exploration and student language before teaching formal math vocabulary and stepwise procedures
• Concrete instruction is not sufficient for relevance
• Use concrete instruction until students are fluent
• Find a way to bridge concrete understanding to abstract computation
## Manipulative Modeling Tips

(Witzel & Allsopp, 2007)

| Use transparent manipulative objects on an overhead projector | Apply magnetic adhesive to a teacher set of manipulative objects to use on magnetic white boards |
| Develop large posterboard renditions of the manipulative objects to use on table tops or wall | Use an elevated table with an angled stand (e.g. chart paper stand) that can support manipulatives securely |
| Move students with visual and attention problems closer to you as you model | Provide students with their own sets of manipulatives to use at their desks |

© Witzel, 2011
Are virtual manips the same as concrete ones?

- Smart boards and Promethean boards
- Example: Virtual Library of Mathematics Manipulatives
  - Look up concrete objects by standard strand and grade level
  - The National Library of Virtual Manipulatives (NLVM) is an NSF supported project that began in 1999 to develop a library of uniquely interactive, web-based virtual manipulatives or concept tutorials, mostly in the form of Java applets, for mathematics instruction (K-12 emphasis). The project includes dissemination and extensive internal and external evaluation.
Manipulative Instruction

• Concrete instructional strategies cannot supplant good teaching practices.

• Benefits for Students:
  1. Develop mathematics concepts.
  2. Develop greater procedural understanding.

• Success leads to confidence. Confidence provides motivation for learning. All students, including students with learning problems, deserve that (Witzel & Allsopp, 2007)
Math Interactions

• **Use language to take students from one level of learning to the next.**

• Ways to increase interactions:
  – Allow students to interact frequently with the class materials and concepts
  – Model and encourage level appropriate math vocabulary in class dialogue
  – Use white boards to assess step by step process of concept
  – Set up cooperative groups with systematic interaction
  – Use journals to practice student think alouds and conceptualization

© Witzel, 2011
## Match instruction to assessment

<table>
<thead>
<tr>
<th></th>
<th>Multi fact</th>
<th>combo</th>
<th>Carry in</th>
<th>Carry out</th>
<th>Add carry</th>
<th>Line up add</th>
<th>Add facts</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mike</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>Tarek</td>
<td>√</td>
<td>X</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>X</td>
</tr>
<tr>
<td>Miguel</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>Manuel</td>
<td>√</td>
<td>√</td>
<td>X</td>
<td>X</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>X</td>
</tr>
<tr>
<td>Jose</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>Pam</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>X</td>
<td>√</td>
<td>X</td>
</tr>
<tr>
<td>Michele</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>Brandon</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>Stan</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>X</td>
<td>√</td>
<td>X</td>
</tr>
</tbody>
</table>
Recap

• What are some necessary foundational skills for algebra success?
• What has been found to constitute effective instruction?
• Why are visual representations so effective?
• What is CRA?
• How will you apply CRA to your own classroom / school?
References


© Witzel, 2011